10/3/19

Last time, we were approximating $S_X(t)$ & $H_X(t)$

$$\left[\ S_X(t) = e^{-H_X(t)}\ \right]$$

Kaplan-Meier Product-Limit Estimate for $S_X(t)$:

$$S_X(t) \approx \prod_{y_j \le t} \left(1 - \frac{s_j}{r_j}\right) = S_n(t)$$

Nelson-Aalen Estimate for $H_X(t)$:

$$H_X(t) \approx \sum_{y_j \le t} \frac{s_j}{r_j} = \hat{H}(t)$$

Examples: (See next pages)

LM.1. An employer is modelling time to retirement of workers using the Kaplan-Meier estimator. You are given the following information.

(i) There are 1000 workers in force at time 0.

(ii) At time 10, there are 600 workers remaining.

(iii) The Kaplan-Meier estimate of the survival function for retirement at time 10 is $\hat{S}_{1000}(10) = 0.8$

(iv) The next retirement after time 10 occurred at time 12, when 100 workers retired.

(v) During the period from time 10 to time 12, a total of 200 workers dropped out for various other reasons.

Calculate $\hat{S}_{1000}(12)$, the Kaplan-Meier estimate of the survival function $S(12)$.

(A) 0.40

(B) 0.45

(C) 0.50

(D) 0.55

(E) 0.60

$$S_{1000}(12) \underset{PL}{\overset{KM}{=}} \prod_{y_j \le 12} \left(1 - \frac{s_j}{r_j}\right)$$

$$= \underbrace{\prod_{y_j \le 10} \left(1 - \frac{s_j}{r_j}\right)}_{= S_{1000}(10) = 0.8} \cdot \left(1 - \frac{s_{t=12}}{r_{t=12}}\right)$$

$s_{t=12} = 100$

In order to proceed, we must assume that (ii) $\Rightarrow$ there are 600 workers remaining after the retirements, if any, at $t=10$.

Then $r_{t=12} = 600 - b_{10 \le t < 12} = 600 - 200 = 400$

$$\therefore S_{1000}(12) = 0.8 \cdot \left(1 - \frac{100}{400}\right) = 0.6$$

LM.3. A cohort of 100 newborns is observed from birth. During the first year, 10 drop out of the study and one dies at time 1. Eight more drop out during the next six months, then, at time 1.5, three deaths occur.

Compute the Nelson-Åalen ~~estimator~~ estimate of the survival function, $S(1.5)$.

(A) 0.950

(B) 0.951

(C) 0.952

(D) 0.953

(E) 0.954

$$\hat{H}(1.5) \stackrel{NA}{=} \sum_{y_j \le 1.5} \frac{s_j}{r_j}$$

y_j	s_j	r_j
1	1	90 = 100 − 10
1.5	3	81 = 90 − 1 died − 8 flew away

$$\therefore \hat{H}(1.5) = \frac{1}{90} + \frac{3}{81} = .0\overline{481}$$

$$\therefore \hat{S}(1.5) = e^{-.0\overline{481}} = 0.95299\ldots$$

$\boxed{D}$

5. You are given the following information derived from a sample of $n = 20$ lives with respect to a study of time-to-failure.

i	y_i	s_i	b_i	r_i
1	1	2	0	20
2	3	3	1	18
3	6	5	2	14
4	8	6	0	

Calculate $\tilde{S}_{20}(y_3)$, the Kaplan-Meier estimate of the survival probability, $S(6)$.

(A) 0.32

(B) 0.35

(C) 0.48

(D) 0.62

(E) 0.75

$$S_{20}(6) \overset{KM}{\underset{PL}{=}} \prod_{y_j \le 6} \left(1 - \frac{s_j}{r_j}\right)$$

$$= \left(1 - \frac{2}{20}\right)\left(1 - \frac{3}{18}\right)\left(1 - \frac{5}{14}\right)$$

$$= .482\ldots$$

Remark: $S_{20}(8) = \left(1-\frac{2}{20}\right)\left(1-\frac{3}{18}\right)\left(1-\frac{5}{14}\right)\left(1-\frac{6}{7}\right) > 0$

$\hookrightarrow 8 = y_{max}$

Note: $S_{20}(t) = .482\ldots$ if $6 \le t < 8$

Q: How do we estimate $S_X(t)$ if $t > y_{max}$?

A: If ~~$\hat{S}_n(t)=0$~~ $\hat{S}_n(y_{max}) = 0$, then $\hat{S}_n(t) = 0$ for all $t > y_{max}$

If $\hat{S}_n(y_{max}) > 0$, then if $t > y_{max}$

1) (Efron) $\Rightarrow$ $\hat{S}_n(t) = 0$

2) (Klein & Moeschberger) $\Rightarrow$ assume there is a terminal age ω such that

$$\hat{S}_n(t) = \begin{cases} \hat{S}_n(y_{max}) & \text{if } t < \omega \\ 0 & \text{if } t \geq \omega \end{cases}$$

3) (Brown, Hollander, & Kowar) $\Rightarrow$

$$\hat{S}_n(t) = \left[\hat{S}_n(y_{max})\right]^{t/y_{max}}$$

Estimating Variances of estimators for $S_X(t)$ using KM & NA.

Notation: Let

V_n = estimated variance of KM estimator of $S_X(t)$

$\hat{V}$ = ——— NA ———

Results:

See the last page of LTAM Tables

$$V_n = [S_n(t)]^2 \cdot \sum_{y_j \le t} \frac{s_j}{r_j(r_j - s_j)} \quad \text{(Greenwood)}$$

$$\hat{V} = [\hat{S}(t)]^2 \cdot \sum_{y_j \le t} \frac{s_j(r_j - s_j)}{r_j^3} \quad \text{(Klein)}$$

There's another one not listed in the tables, but on the syllabus for L-TAM:

$$V = [\hat{S}(t)]^2 \cdot \sum_{y_j \le t} \frac{s_j}{r_j^2} \quad \text{(Aalen)}$$

Note: Aalen $\Rightarrow$ replace $r_j - s_j$ in Klein by r_j